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On a discretization of asymptotic nets☆

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Abstract

We discretize some notions of the theory of asymptotic nets and of the theory of transformations of asymptotic nets. These are the Lelieuvre formulas, the Moutard equation, the Moutard transformation, the Weingarten congruences and the Jonas formulas. It allows us to extend the theory of reductions of the discrete version of the Darboux system, applied primarily to multidimensional quadrilateral lattices, on the theory of discrete asymptotic nets which in turn is helpful in a discretization of some classical differential nonlinear integrable systems of physical interest, e.g. the Ernst equation and the stationary modified Nizhnik–Veselov–Novikov system (in form which we call the Fubini–Ragazzi system). © 2002 Published by Elsevier Science B.V.

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1. Introduction

To a given nonlinear completely integrable system of differential equations $\mathcal{S}$ one can relate many systems of difference equations which become the differential system when appropriate limiting process is applied. It is natural task to isolate (from this set of systems of difference equations) a system of difference equations $\mathcal{DS}$ that exhibits the integrability features like existence of a Lax pair, a Darboux–Bäcklund transformation and a permutability theorem. Such a system $\mathcal{DS}$ of difference equations we call *a discrete version* of the given system $\mathcal{S}$.

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The discrete version of the classical Darboux system [11,13,18,19], which is often regarded as a master equation of the theory of nonlinear integrable systems, has been proposed recently [8]. Moreover, the theory of integrable reductions of the Darboux system [49,50], which contains the theory of reductions of transformations of the Laplace equation, is systematically being extended on the discrete case [10,13,17,32,36] and our paper belongs to this stream. It was shown in [15] that conjugate nets (see, e.g. [11,13,18,19]) governed by the Darboux system have a discrete counterpart, the so-called multidimensional quadrilateral lattices governed by the discrete version of the Darboux system. The theory of integrable reductions of the discrete version of the Darboux system (and transformations of the Laplace equation) has been applied to the multidimensional quadrilateral lattices [10,13,17,32], but the theory can be applied to geometric objects different from conjugate nets and their discrete counterparts (multidimensional quadrilateral lattices) for instance to asymptotic nets and their discrete counterparts (the definition of asymptotic nets and discrete asymptotic nets will be given in Section 2). That is why we find interesting to discretize some notions of the classical differential geometry which concerns the asymptotic nets. The main result of the paper is a discretization of Jonas formulas [27] (Sections 9 and 10) which in turn is helpful in a discretization of some classical integrable systems of physical interest.

Our goal is to discuss these aspects of (discrete) asymptotic nets that are responsible for integrable phenomena. Therefore besides the Jonas formulas we discuss in Sections 3–6 a discretization of the Moutard transformation [40] and Lelieuvre formulas [31] (Theorems 3 and 4 and Remark 5 develop some ideas of the papers [31,40]) and also we introduce in Section 8 the notion of discrete tangent canonical fields of a discrete asymptotic net. Two special classes of the discrete asymptotic nets have been widely discussed so far. These are discrete analogs of surfaces with constant negative curvature (the so-called K-surfaces, see [5,16] and references therein) and discrete analogs of affine spheres (see [7,6,45]).

To show you our physical motivation let us recall that it was considerations on asymptotic nets and their transformations which reveal (among others) the following integrable systems.

1.1. Fubini–Ragazzi system

By the Fubini–Ragazzi (or isothermally asymptotic [22,24,29,41]) system, which can be related (see [21]) to either modified stationary Nizhnik–Veselov–Novikov or stationary Nizhnik–Veselov–Novikov (see also [25,30,37,46,48]), we understand the following system:

$$
\begin{aligned}
&(-a_{,u} + \tfrac{1}{2}a^2 + pb + p_{,v})_{,v} = 2pq_{,u} + qp_{,u},\\
&(-b_{,v} + \tfrac{1}{2}b^2 + qa + q_{,u})_{,u} = 2qp_{,v} + pq_{,v},\\
&a_{,v} = b_{,u}, \qquad \left(\log \frac{p}{q}\right)_{,uv} = 0, \qquad (1)
\end{aligned}
$$

where p, q, a and b are real functions of both parameters u and v. This system was primarily introduced by Fubini [23]. A Darboux–Bäcklund transformation and superposition principle were obtained by Ragazzi in the paper [41] (see also [22,24,29]). This justifies our terminology.

The system (1) is of interest for, as we have already mentioned it is related to the stationary modified Nizhnik–Veselov–Novikov system, secondly, it is a generalization of the Tzitzeica (Dodd–Bullough) equation. Indeed, on rescaling the parameters $\tilde{u} = f(u)$, $\tilde{v} = g(v)$ (the tilde will be omitted) one can reduce the last equation of Eq. (1) to the condition

$$p = q, \tag{2}$$

due to third equation of (1) we can introduce a potential ϑ such that

$$a = \vartheta_{,u} \qquad b = \vartheta_{,v}, \tag{3}$$

and then the reduction

$$p = \mathrm{e}^{-\vartheta} \tag{4}$$

cause that system (1) is reducible to the Tzitzeica equation

$$\vartheta_{,uv} = c\,\mathrm{e}^{\vartheta} - \mathrm{e}^{-2\vartheta}, \tag{5}$$

where c is a constant of integration. The discrete version of the Tzitzeica equation (5) has been proposed in [45] (cf. [6,7]). An intriguing link between the Tzitzeica equation (discrete version of Tzitzeica equation) and self-dual Einstein spaces via permutability theorem for the Tzitzeica equation (discrete version of Tzitzeica equation) was established in [44,45].

The considerations contained in the present paper have allowed us to find a discrete version of the Fubini–Ragazzi system. A detailed discussion of the discrete version of the Fubini–Ragazzi system is beyond the frames of this article. However, in Section 11, we argue that the discrete version of (1) is

$$\begin{aligned} &\frac{A_{(22)}}{AH_{(2)}} = \frac{B_{(11)}}{BH_{(1)}}, \qquad \frac{A_{(22)}H}{A_{(2)}H_{(2)}}(1 + B - Q) + Q_{(1)}C_{(2)} - D_{(1)} = 0,\\ &\frac{B_{(11)}H}{B_{(1)}H_{(1)}}(1 + A - P) + P_{(2)}D_{(1)} - C_{(2)} = 0,\\ &H_{(2)}\left(\frac{B_{(1)}}{B}\frac{A}{A_{(2)}}\right)_{(12)} \diamond P_{(2)} = H_{(1)}\left(\frac{B}{B_{(1)}}\frac{A_{(2)}}{A}\right)_{(12)} \diamond Q_{(1)}, \end{aligned} \tag{6}$$

where a subscript in brackets denotes the increment of the parameter, for instance $f(m_1, m_2)_{(1)} = f(m_1 + 1, m_2)$, $f(m_1, m_2)_{(2)} = f(m_1, m_2 + 1)$, $f(m_1, m_2)_{(12)} = f(m_1 + 1, m_2 + 1)$, $f(m_1, m_2)_{(-1)} = f(m_1 - 1, m_2)$, etc., the symbol $\diamond$ stands for the following operator:

$$\diamond f := \frac{f_{(12)} f}{f_{(1)} f_{(2)}},$$

and the functions C and D are defined by

$$C := 1 + \frac{A_{(2)}}{H} + \frac{B_{(1)}P_{(2)}}{H}, \qquad D := 1 + \frac{B_{(1)}}{H} + \frac{A_{(2)}Q_{(1)}}{H}, \tag{7}$$

while H is given by

$$H := 1 - P_{(2)}Q_{(1)}. \tag{8}$$

1.2. Bianchi–Ernst system

The Euclidean Bianchi–Ernst system ($\epsilon = 1$) or *the Minkowski Bianchi–Ernst system* ($\epsilon = -1$) is the following system of differential equations on real functions N_0, N_1, N_2 of real parameters u and v:

$$\frac{(N_0)_{,uv}}{N_0} = \frac{(N_1)_{,uv}}{N_1} = \frac{(N_2)_{,uv}}{N_2}, \tag{9}$$

$$\hat{N} \cdot \hat{N} = U(u) + V(v), \tag{10}$$

where $\hat{N} = (N_0, N_1, N_2)$ and dot stands for the scalar product

$$\hat{N} \cdot \hat{N} := (N_0)^2 + \epsilon((N_1)^2 + (N_2)^2).$$

The system (more precisely its Euclidean case) was introduced by Bianchi in [2,3] (see also [39]) and describes the so-called Bianchi surfaces, a generalization of mentioned K-surfaces. It was Bianchi, who found a Darboux–Bäcklund transformation and superposition principle for solutions of the system (9) and (10). That is why the system takes its name after Bianchi. The name of Ernst appears due to the fact that in the Minkowski case ($\epsilon = -1$) on defining $n_i = N_i/\sqrt{r}$ (where $r = \hat{N} \cdot \hat{N}$), making stereo-graphical projection

$$\xi = \frac{n_1 + \mathrm{i}n_2}{1 + n_0}, \tag{11}$$

and finally changing independent variables

$$u = \rho + \mathrm{i}z, \qquad v = \rho - \mathrm{i}z, \tag{12}$$

we come to the Ernst equation

$$(\xi\bar{\xi} - 1)\left(\xi_{,\rho\rho} + \xi_{,zz} + \frac{r_{,\rho}}{r}\xi_{,\rho} + \frac{r_{,z}}{r}\xi_{,z}\right) = 2\bar{\xi}((\xi_{,\rho})^2 + (\xi_{,z})^2), \quad r_{,\rho\rho} + r_{,zz} = 0. \tag{13}$$

This observation would appear in several papers [33,43,47] just after the result by Bianchi was revised [9,35]. Let us recall that every solution to Ernst equation describes an axisymmetric stationary vacuum Einstein field [20]. This is another reason why we find interesting to take up the subject of discretization of asymptotic nets (it yields hints toward the discretization of the Bianchi–Ernst system).

A discrete version of the system (9) and (10) (as it will be shown in the forthcoming paper [14]) is the system on real functions N_0, N_1, N_2 of integer parameters m_1 and m_2

$$\frac{N_{0(12)} + N_0}{N_{0(1)} + N_{0(2)}} = \frac{N_{1(12)} + N_1}{N_{1(1)} + N_{1(2)}} = \frac{N_{2(12)} + N_2}{N_{2(1)} + N_{2(2)}}, \tag{14}$$

$$(\hat{N}_{(12)} + \hat{N}) \cdot (\hat{N}_{(1)} + \hat{N}_{(2)}) = 4U(m_1) + 4V(m_2). \tag{15}$$

2. Asymptotic nets in the affine space A^3 and their discretization

We start from basic definitions. By *a regular net* in the affine space A^3 we understand the image $\mathcal{N}$ of an open subset U of $\mathbb{R}^2$ under the diffeomorphism

$$x:\mathbb{R}^2\supset U\to\mathcal{N}\subset A^3,\qquad U\ni(u,v)\overset{x}{\mapsto}p\in\mathcal{N} \tag{16}$$

together with a natural structure of lines distinguished by this map, i.e. with two one-parameter families of curves the first one consisting of the curves $u=$ const. and the second one consisting of the curves $v=$ const. The curves we shall call *the lines of the net*. A regular net we call an *asymptotic net* if at every point p of the net we have: osculating planes at p of the two lines of the net that pass through p coincide. It means that at every point of the asymptotic net we have (in the whole paper $\vec{r}(u,v)$ stands for the radius vector of the net)

$$\vec{r}_{,uu}=a\vec{r}_{,u}+p\vec{r}_{,v},\qquad \vec{r}_{,vv}=q\vec{r}_{,u}+b\vec{r}_{,v}. \tag{17}$$

From the equality $\vec{r}_{,uuvv}=\vec{r}_{,vvuu}$, we obtain that the functions a,b,p and q satisfy (provided that at every point of the net $\{\vec{r}_{,u};\vec{r}_{,v};\vec{r}_{,uv}\}$ is a basis) the differential constraints (compatibility conditions)

$$\begin{aligned}&(-a_{,u}+\tfrac{1}{2}a^2+pb+p_{,v})_{,v}=2pq_{,u}+qp_{,u},\\&(-b_{,v}+\tfrac{1}{2}b^2+qa+q_{,u})_{,u}=2qp_{,v}+pq_{,v},\\&a_{,v}=b_{,u}\Rightarrow a=\vartheta_{,u},\quad b=\vartheta_{,v}.\end{aligned} \tag{18}$$

Since Eqs. (17) are invariant with respect to affine transformations of the space they can be used to discuss the asymptotic nets in the affine geometry.

By a *regular discrete net* in A^3, we understand the image of $\mathbb{Z}^2$ under the map

$$x:\mathbb{Z}^2\to A^3,\qquad \mathbb{Z}^2\ni(m_1,m_2)\mapsto p\in A^3 \tag{19}$$

together with:

- structure of discrete lines distinguished by this map, i.e. lines $m_1=$ const. and $m_2=$ const. the so-called *discrete lines* of the discrete net;
- a regularity condition: points $\vec{R}$, $\vec{R}_{(1)}$ and $\vec{R}_{(2)}$ are not collinear ($\vec{R}(m_1,m_2)$ stands for radius vector of the discrete net).

A discrete net we call *a discrete asymptotic net* (see *Schmiegliniennetze* in [42]) if the plane through the points $\vec{R}(m_1,m_2)_{(-1)}$, $\vec{R}(m_1,m_2)$, $\vec{R}(m_1,m_2)_{(1)}$ (the osculating plane to a line from the first family of lines at the point $\vec{R}(m_1,m_2)$) coincides with the plane through the points $\vec{R}(m_1,m_2)_{(-2)}$, $\vec{R}(m_1,m_2)$, $\vec{R}(m_1,m_2)_{(2)}$ (the osculating plane to a line from the second family of lines at the point $\vec{R}(m_1,m_2)$) at every point $\vec{R}(m_1,m_2)$ of the net. Note that discrete asymptotic nets are the exceptional case (among other discrete nets) when a tangent plane to the point of the net is well defined.

From the definition of discrete asymptotic nets we can write

$$\begin{aligned}&\vec{R}_{(11)}-\vec{R}_{(1)}=A(\vec{R}_{(1)}-\vec{R})+P(\vec{R}_{(12)}-\vec{R}_{(1)}),\\&\vec{R}_{(22)}-\vec{R}_{(2)}=B(\vec{R}_{(2)}-\vec{R})+Q(\vec{R}_{(12)}-\vec{R}_{(2)})\end{aligned} \tag{20}$$

or

$$\vec{R}_{(112)} - \vec{R}_{(12)} = \frac{A_{(2)}}{H}(\vec{R}_{(12)} - \vec{R}_{(2)}) + \frac{P_{(2)}B_{(1)}}{H}(\vec{R}_{(12)} - \vec{R}_{(1)}),$$
$$\vec{R}_{(221)} - \vec{R}_{(12)} = \frac{B_{(1)}}{H}(\vec{R}_{(12)} - \vec{R}_{(1)}) + \frac{Q_{(1)}A_{(2)}}{H}(\vec{R}_{(12)} - \vec{R}_{(2)}), \qquad (21)$$

where H is given by Eq. (8). The equality $\vec{R}_{(1122)} = \vec{R}_{(2211)}$ gives the following compatibility conditions (provided that $\{(\vec{R}_{(1)} - \vec{R}), (\vec{R}_{(2)} - \vec{R}), (\vec{R}_{(12)} - \vec{R})\}$ form a basis at every point of the net)

$$\frac{A_{(22)}}{AH_{(2)}} = \frac{B_{(11)}}{BH_{(1)}}, \qquad \frac{A_{(22)}H}{A_{(2)}H_{(2)}}(1 + B - Q) + Q_{(1)}C_{(2)} - D_{(1)} = 0,$$
$$\frac{B_{(11)}H}{B_{(1)}H_{(1)}}(1 + A - P) + P_{(2)}D_{(1)} - C_{(2)} = 0, \qquad (22)$$

where C and D are defined by Eq. (7).

3. Lelieuvre formulas. Asymptotic nets in equi-affine space eA^3

Let us enrich the affine space with volume form *Vol* (by *Vol**, we denote the dual form of *Vol*). It enable us to construct cross-product from ordered pair of linearly independent vectors (say $(\vec{a}, \vec{b})$), i.e. element $\hat{N} \in T^*eA^3$ such that $\langle \hat{N}|\vec{a}\rangle = 0 = \langle \hat{N}|\vec{b}\rangle$ and $\langle \hat{N}|\vec{c}\rangle = Vol\{\vec{a}; \vec{b}; \vec{c}\}$ for every $\vec{c} \in TeA^3$. The following theorem provides a linear procedure of constructing asymptotic nets.

Theorem 1 (Lelieuvre [34]).

1. *If an asymptotic net $\vec{r}(u, v)$ is given and if a conormal field $\hat{n}(u, v)$ to the net respects condition $Vol^*(\hat{n}; \hat{n}_{,u}; \hat{n}_{,v}) \neq 0$ then there exists a conormal field $\hat{N}(u, v)$ such that*

$$\hat{N}_{,uv} = f(u, v)\hat{N}, \qquad (23)$$

$$\vec{r}_{,u} = \hat{N}_{,u} \times \hat{N}, \qquad \vec{r}_{,v} = \hat{N} \times \hat{N}_{,v}. \qquad (24)$$

2. *Let $\hat{N} = [N_0, N_1, N_2]$ is a solution to* (23) *obeying $Vol^*(\hat{N}; \hat{N}_{,u}; \hat{N}_{,v}) \neq 0$. Then $\vec{r}(u, v)$ obtained from* (24) *is a radius vector of a regular asymptotic net.*

The formulas (24) are called Lelieuvre formulas. The Lelieuvre formulas are invariant with respect to equi-affine transformations of the space and therefore can be used to discuss asymptotic nets in equi-affine geometry.

On using the Lelieuvre formulas Eqs. (17) and (23) take the form

$$\hat{N}_{,uu} = a\hat{N}_{,u} - p\hat{N}_{,v} + \gamma\hat{N}, \qquad \hat{N}_{,vv} = -q\hat{N}_{,u} + b\hat{N}_{,v} + \delta\hat{N}, \quad \hat{N}_{,uv} = f\hat{N}. \qquad (25)$$

Compatibility conditions of (25) are Eq. (18) together with

$$\gamma = p_{,v} + p\vartheta_{,v}, \qquad \delta = q_{,u} + q\vartheta_{,u}, \qquad f = \vartheta_{,uv} + pq. \tag{26}$$

4. The Moutard equation and the Moutard transformation

In connection with the previous section we pay our attention to the equation

$$N_{,uv} = FN, \tag{27}$$

which is called *the Moutard equation* and is a member of a class of the Laplace equations with the equal Laplace invariants (see, e.g. [1]). From the point of view of integrable systems the most important is that the form of the equation is covariant under the following *Moutard transformation* [38]. Let a function Θ will be given. Then transformation $N \to N'$ given by

$$(N'\Theta)_{,u} = N\Theta_{,u} - N_{,u}\Theta, \qquad (N'\Theta)_{,v} = -N\Theta_{,v} + N_{,v}\Theta \tag{28}$$

is an invertible map between the solution space of Eq. (27) and the solution space of an equation

$$N'_{,uv} = f'N', \tag{29}$$

where

$$f = \frac{\Theta_{,uv}}{\Theta}, \qquad f' = \frac{(1/\Theta)_{,uv}}{1/\Theta}. \tag{30}$$

If we take three solutions $[N_0, N_1, N_2]$ to (27), we obtain after a Moutard transformation a three solution $[N'_0, N'_1, N'_2]$ to (29). Owing to the Lelieuvre formulas we receive a transformation between asymptotic nets (cf. Section 7).

The question arises what is a discrete version of the Moutard equation? The permutability theorem for the Moutard transformations in its classical form [4,22] reinterpreted as a integrable difference equation (see [45] and references therein) suggests that a discrete counterpart of the Moutard equation is

$$N_{(12)} - N = f(N_{(1)} - N_{(2)}), \tag{31}$$

and in the paper [40] Eq. (31) is treated as a discrete version of the Moutard equation. But we can straightforward modify (as it was suggested to me by Doliwa) the classical Moutard transformation.

Theorem 2. *Let a solution of the Moutard equation* (27) *N and its two Moutard transforms: the first one denoted by* $N^{(1)}$ (*superscript instead of prime* N' *in formulas* (28)!) *governed by* (28) *with function* $\Theta^{(1)}$ *instead of function* Θ *and the second one denoted by* $N^{(2)}$ *governed by* (28) *with mutually interchanged parameters* $u \leftrightarrow v$, *with function* $\Theta^{(2)}$ *instead of function* Θ, *are given. Then function* $N^{(12)}$ *given by*

$$N^{(12)} = -N + \frac{\Theta^{(1)}\Theta^{(2)}}{\nu}(N^{(2)} + N^{(1)}), \tag{32}$$

where ν is given by the quadratures

$$\nu_{,u} = \Theta^{(2)}\Theta^{(1)}{}_{,u} - \Theta^{(1)}\Theta^{(2)}{}_{,u}, \qquad \nu_{,v} = -\Theta^{(2)}\Theta^{(1)}{}_{,v} + \Theta^{(1)}\Theta^{(2)}{}_{,v}$$

is solution of Moutard equation

$$N^{(12)}{}_{,uv} = f^{(12)}N^{(12)},$$

where

$$f^{(12)} = f + \nu\left(\frac{1}{\nu}\right)_{,uv} + \frac{1}{\nu}(\Theta^{(2)}{}_{,u}\Theta^{(1)}{}_{,v} - \Theta^{(1)}{}_{,u}\Theta^{(2)}{}_{,v}).$$

The proof is analogous to the classical one. So Eq. (32) suggests that the Moutard equation is

$$N_{(12)} + N = F(N_{(2)} + N_{(1)}). \tag{33}$$

As we shall see either Eq. (31) or Eq. (33) can be treated as a discrete version of the Moutard equation and both are useful in the construction of discrete asymptotic nets.

5. Discrete Moutard transformation and discrete Moutard equation

We start our discrete considerations from recalling the *discrete Moutard transformation* which one can find in slightly modified form (suitable for Eq. (31)) in [40]. Namely, one can by direct calculation see that transformation $N \to N'$ given by

$$\Delta_{(1)}(N'\Theta) = N\Delta_{(1)}\Theta - \Theta\Delta_{(1)}N, \qquad \Delta_{(2)}(N'\Theta) = \Theta\Delta_{(2)}N - N\Delta_{(2)}\Theta \tag{34}$$

(where $\Delta_{(i)}f = f_{(i)} - f$) maps from the solution space of a *discrete Moutard equation* (33) into solution space of another discrete Moutard equation

$$N'_{(12)} + N' = F'(N'_{(1)} + N'_{(2)}), \tag{35}$$

where

$$F = \frac{\Theta_{(12)} + \Theta}{\Theta_{(1)} + \Theta_{(2)}}, \qquad F' = \frac{(1/\Theta)_{(12)} + 1/\Theta}{(1/\Theta)_{(1)} + (1/\Theta)_{(2)}}. \tag{36}$$

So we have to our disposal a transformation which exhibits integrable features.

We can introduce for the discrete Laplace equation $\psi_{(12)} + \alpha\psi_{(1)} + \beta\psi_{(2)} + \gamma\psi = 0$ ($L\psi = 0$ for short) which is form-invariant under the gauge $\psi \to (1/A_{(12)})L(A\psi)$ the following invariants of the gauge:

$$h = \frac{\alpha_{(12)}}{\alpha}\frac{\gamma_{(1)}}{\gamma_{(12)}}, \qquad k = \frac{\beta_{(12)}}{\beta}\frac{\gamma_{(2)}}{\gamma_{(12)}}.$$

The choice of invariants was inspired by the fact that gauge independent characterization of the discrete Moutard equation is $k = h$ just like in the continuous case. Either Eq. (31)

or Eq. (33) satisfy this condition. The discrete version of the theorem (2) (independently found in [12]) is the following theorem.

Theorem 3. *Let a solution of discrete Moutard equation* (33) *N and its two transforms: the first one denoted by* $N^{(1)}$ *governed by* (34) *with function* $\Theta^{(1)}$ *instead of* Θ *and the second one denoted by* $N^{(2)}$ *governed by* (34) *with mutually interchanged parameters* ${}_{(1)} \leftrightarrow {}_{(2)}$ *with function* $\Theta^{(2)}$ *instead of* Θ*, are given. Then function* $N^{(12)}$ *obtained from*

$$N^{(12)} = -N + \frac{\Theta^{(1)}\Theta^{(2)}}{\nu}(N^{(2)} + N^{(1)}), \tag{37}$$

where ν *is given by*

$$\Delta_{(1)}\nu = \Theta^{(2)}\Theta^{(1)}_{(1)} - \Theta^{(1)}\Theta^{(2)}_{(1)}, \qquad \Delta_{(2)}\nu = -\Theta^{(2)}\Theta^{(1)}_{(2)} + \Theta^{(1)}\Theta^{(2)}_{(2)}$$

is solution of the discrete Moutard equation

$$N^{(12)}_{(12)} + N^{(12)} = F^{(12)}(N^{(12)}_{(1)} + N^{(12)}_{(2)}),$$

where

$$F^{(12)} = \frac{\nu_{(1)}\nu_{(2)}}{\nu_{(12)}\nu}F.$$

6. Discrete Lelieuvre formulas

Let us consider a vector $\hat{N} = [N_0, N_1, N_2]$ of T^*eA^3 components of which satisfy the discrete Moutard equation (33)

$$\hat{N}_{(12)} + \hat{N} = F(\hat{N}_{(1)} + \hat{N}_{(2)}). \tag{38}$$

Hence, we have

$$(\hat{N}_{(12)} + \hat{N}) \times (\hat{N}_{(1)} + \hat{N}_{(2)}) = 0, \tag{39}$$

and after some calculations we obtain

$$\Delta_{(2)}(\Delta_{(1)}\hat{N} \times \hat{N}) = \Delta_{(1)}(\hat{N} \times \Delta_{(2)}\hat{N}). \tag{40}$$

From the above we infer that there exists vector $\vec{r}$ such that

$$\Delta_{(1)}\vec{r} = \Delta_{(1)}\hat{N} \times \hat{N}, \qquad \Delta_{(2)}\vec{r} = \hat{N} \times \Delta_{(2)}\hat{N}. \tag{41}$$

In analogy to the continuous case we call Eq. (41) discrete Lelieuvre formulas for as it is easy to show the vector $\vec{r}$ can be interpreted as a radius vector of a discrete asymptotic net. Indeed if $\vec{r}$ describes a net obtained from (41) (it is assumed that (38) holds) then $\langle\hat{N}|\Delta_{(1)}\vec{r}\rangle = 0 = \langle\hat{N}|\Delta_{(2)}\vec{r}\rangle$ and in this sense $\hat{N}$ is conormal to the net. For any field $\hat{n}$

proportional to $\hat{N}$, we have

$$\langle \hat{n}|\Delta_{(1)}\vec{r}\rangle = 0, \qquad \langle \hat{n}|\Delta_{(2)}\vec{r}\rangle = 0, \qquad \langle \Delta_{(1)}\hat{n}|\Delta_{(1)}\vec{r}\rangle = 0, \quad \langle \Delta_{(2)}\hat{n}|\Delta_{(2)}\vec{r}\rangle = 0. \tag{42}$$

The conditions (42) are equivalent to condition that $\vec{r}, \vec{r}_{(1)}, \vec{r}_{(2)}, \vec{r}_{-(1)}, \vec{r}_{-(2)}$ are coplanar and hence the discrete net is asymptotic. Inversely, if a discrete asymptotic net is given then one can find the discrete conormal field $\hat{N}$ such that (41) holds. Indeed, if a discrete asymptotic net is given then (42) holds and we infer that $\Delta_{(1)}\vec{r} = s\Delta_{(1)}\hat{n} \times \hat{n}$ and $\Delta_{(2)}\vec{r} = t\hat{n} \times \Delta_{(2)}\hat{n}$. From the equalities $\langle \hat{n}|\Delta_{(1)}\Delta_{(2)}\vec{r}\rangle = \langle \hat{n}|\Delta_{(2)}\Delta_{(1)}\vec{r}\rangle$ and $\langle \hat{n}_{(12)}|\Delta_{(1)}\Delta_{(2)}\vec{r}\rangle = \langle \hat{n}_{(12)}|\Delta_{(2)}\Delta_{(1)}\vec{r}\rangle$, we infer that

$$s_2 s = t_1 t \tag{43}$$

provided that $Vol^*(\hat{n}_{(1)}; \hat{n}_{(12)}; \hat{n}) \neq 0 \neq Vol^*(\hat{n}_{(2)}; \hat{n}_{(12)}; \hat{n})$. It means that there exists potential r such that $s = r_{(1)}r$ and $t = r_{(2)}r$. On introducing $\hat{N} = r\hat{n}$ we come to formulas (41). Hence we have the following theorem.

Theorem 4 (Discrete Lelieuvre formulas).

1. *If the asymptotic net $\vec{r}$ is given and if a discrete conormal field of the net $\hat{n}$ respect the conditions $Vol^*(\hat{n}_{(1)}; \hat{n}_{(12)}; \hat{n}) \neq 0 \neq Vol^*(\hat{n}_{(2)}; \hat{n}_{(12)}; \hat{n})$ then there exist a conormal field $\hat{N}$ such that Eqs.* (41) *and* (38) *hold.*
2. *Let $\hat{N} = [N_0, N_1, N_2]$ is solution to* (38) *obeying $Vol^*(\hat{N}_{(1)}; \hat{N}_{(12)}; \hat{N}) \neq 0 \neq Vol^*(\hat{N}_{(2)}; \hat{N}_{(12)}; \hat{N})$. Then $\vec{r}$ obtained from* (41) *is the radius vector of a discrete asymptotic net.*

Remark 5. Since Eq. (43) is quadratic we have an alternative way of description of discrete asymptotic nets. Namely, we can take $s = r_{(1)}r$ and $t = -r_{(2)}r$. Then the discrete Lelieuvre formulas take the form

$$\Delta_{(1)}\vec{r} = \Delta_{(1)}\hat{N} \times \hat{N}, \qquad \Delta_{(2)}\vec{r} = \Delta_{(2)}\hat{N} \times \hat{N}, \tag{44}$$

while the Moutard equation is

$$\hat{N}_{(12)} - \hat{N} = \tilde{F}(\hat{N}_{(1)} - \hat{N}_{(2)}). \tag{45}$$

Note that formulas (41) have more natural continuous limit than formulas (44).

Using discrete Lelieuvre formulas (41), we can rewrite Eqs. (20) and (38) as

$$\begin{aligned} &\hat{N}_{(11)} - \hat{N}_{(1)} = A(\hat{N}_{(1)} - \hat{N}) - P(\hat{N}_{(12)} - \hat{N}_{(1)}) + \gamma N_1, \\ &\hat{N}_{(22)} - \hat{N}_{(2)} = B(\hat{N}_{(2)} - \hat{N}) - Q(\hat{N}_{(12)} - \hat{N}_{(2)}) + \delta N_2, \\ &\hat{N}_{(12)} + N = F(\hat{N}_{(1)} + \hat{N}_{(2)}). \end{aligned} \tag{46}$$

Compatibility conditions of (46) give Eq. (22) together with

$$FF_{(1)} = \frac{A_{(2)}}{AH}, \qquad FF_{(2)} = \frac{B_{(1)}}{BH}, \qquad \gamma = -1 - A - P + \frac{C}{F_{(1)}},$$
$$\delta = -1 - B - Q + \frac{D}{F_{(2)}}. \tag{47}$$

7. Weingarten congruences and their discretization

By a (*rectilinear*) *congruence* in A^3 parameterized by coordinates (u, v), we understand image $\mathcal{K}$ of an open subset U of $\mathbb{R}^2$ under the diffeomorphism

$$x : \mathbb{R}^2 \supset U \to \mathcal{K} \subset LA^3, \qquad U \ni (u, v) \mapsto p \in \mathcal{K}, \tag{48}$$

where LA^3 denotes the set of all straight lines of A^3. We restrict ourselves to congruences such that for every $p \in \mathcal{K}$ there exists exactly two developable surfaces built of the lines of the congruence which pass through p. The curves of regression of the developable surfaces lie on two *focal surfaces*. A map between the two focal surfaces via lines of the congruence is called *focal map* (cf. [22]).

Remark 6. By definition lines of a congruence are tangent to both focal surfaces.

An asymptotic net that lie on a focal surface of a congruence we call *an asymptotic focal net*. A congruence for which an asymptotic focal net under the focal map is an asymptotic focal net again is called *Weingarten* (*or W-*)*congruences*. We use only asymptotic parameterization of W-congruence, i.e. parameters (u, v) imprint asymptotic nets on focal surfaces of the W-congruence. We have the following theorem.

Theorem 7 (Guichard [26,51]).

1. *Let an asymptotic net $\mathcal{N}$ is given and let $\mathcal{N}'$ is asymptotic net obtained from $\mathcal{N}$ via a Moutard transformation. Then there is a rigid motion of $\mathcal{N}'$ such that $\mathcal{N}$ and $\mathcal{N}'$ become asymptotic focal nets of a W-congruence.*
2. *Let a W-congruence is given. There exist a Moutard transformation which maps one asymptotic focal net of the W-congruence to the second asymptotic focal net of the W-congruence.*

We refer to [22] for proof.

On the discrete level one can define a *discrete congruence* in A^3 by injective map

$$x : \mathbb{Z}^2 \to LA^3, \qquad \mathbb{Z}^2 \ni (m_1, m_2) \mapsto p \in LA^3, \tag{49}$$

but is there an analog of focal nets, is this the right analog of continuous congruence? Doliwa and Santini [15] proposed to add condition that two consecutive lines of the congruence intersect. This allowed them to establish a discrete analog of mapping between conjugate nets and it seemed to be discrete analog of focal mapping. In what follows we propose the notion of discrete Weingarten (or W-)congruences which does not fit to Doliwa and Santini scheme (the discrete W-congruences are not special case of the congruences proposed by Doliwa and Santini). The crucial observation is the following remark.

Remark 8. Let us replace Θ and $\hat{N}'$ in formulas (34) by $\Theta^{(1)}$ and $\hat{N}^{(1)}$ (superscript!), respectively. Let us replace Θ and $\hat{N}'$ in formulas (34) with interchanged indices $_{(1)} \leftrightarrow _{(2)}$

by $\Theta^{(2)}$ and $\hat{N}^{(2)}$, respectively. We also introduce operator $\Delta^{(i)}$ by $\Delta^{(i)} f = f^{(i)} - f$. Then a consequence of the discrete Moutard transformation (34) is

$$\Delta^{(1)}\vec{r} = \Delta^{(1)}\hat{N} \times \hat{N} + \vec{c^1}, \qquad \Delta^{(2)}\vec{r} = \hat{N} \times \Delta^{(2)}\hat{N} + \vec{c^2}, \tag{50}$$

where $\vec{c}^1$ and $\vec{c}^2$ are constant vectors.

One can easily achieve above result by the subsequent cross multiplication of both sides of discrete Moutard transformation (34) by $\hat{N}$, $\hat{N}_{(1)}$, $\hat{N}_{(2)}$, $\hat{N}^{(1)}$ and $\hat{N}^{(2)}$. From equations obtained in this way one can infer $\Delta_{(1)}\Delta^{(1)}\vec{r} = \Delta_{(1)}(\Delta^{(1)}\hat{N} \times \hat{N})$, $\Delta_{(2)}\Delta^{(1)}\vec{r} = \Delta_{(2)}(\Delta^{(1)}\hat{N} \times \hat{N})$, $\Delta_{(1)}\Delta^{(1)}\vec{r} = -\Delta_{(1)}(\Delta^{(2)}\hat{N} \times \hat{N})$ and $\Delta_{(2)}\Delta^{(1)}\vec{r} = -\Delta_{(2)}(\Delta^{(2)}\hat{N} \times \hat{N})$ and in result (50).

By rigid motion of transformed nets $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$ we can achieve $\vec{c}^1 = 0 = \vec{c}^2$

$$\Delta^{(1)}\vec{r} = \Delta^{(1)}\hat{N} \times \hat{N}, \qquad \Delta^{(2)}\vec{r} = \hat{N} \times \Delta^{(2)}\hat{N}. \tag{51}$$

The vector $\vec{r}^{(1)} - \vec{r}$ lies in the tangent planes of net $\mathcal{N}$ and $\mathcal{N}^{(1)}$ at points $\vec{r}$ and $\vec{r}^{(1)}$. So, by analogy to the continuous case we define discrete W-congruences.

Proposition 9 (Discrete W-congruences). *Let two radius vectors of discrete asymptotic nets $\vec{r}$ and $\vec{r}^{(i)}$ are related by one of the formulas* (51) ($i = 1$ *or* 2). *A discrete congruence that every line $l(m_1, m_2)$ passes through the points $\vec{r}(m_1, m_2)$ and $\vec{r}^{(i)}(m_1, m_2)$ is called discrete Weingarten* (*or W-*)*congruence*.

On applying permutability theorem for discrete Moutard transformation to the discrete conormal field $\hat{N}$, we build from a given point $\vec{r}$ an asymptotic net (superscripts are thought to be shifts). Indeed, from (37) we get $(\hat{N}^{(12)} + \hat{N}) \times (\hat{N}^{(1)} + \hat{N}^{(2)}) = 0$ and (51) are just Lelieuvre formulas.

Finally, we would like to mention that rectilinear congruences as two-parameter subsets of line geometry are represented by surfaces (nets) in the Plücker–Klein quadric. W-congruences parameterized asymptotically are represented by conjugate nets in the Plücker–Klein quadric [19]. In the paper [12] it was proved that discrete W-congruences proposed in our paper are represented by two-dimensional quadrilateral lattices in the Plücker–Klein quadric.

8. Tangent canonical fields and their discretization

Let us introduce *the tangent canonical fields* $\vec{W}$ and $\vec{Z}$ of the asymptotic net

$$\vec{W} = \mathrm{e}^{-\vartheta}\vec{r}_{,u}, \qquad \vec{Z} = \mathrm{e}^{-\vartheta}\vec{r}_{,v}. \tag{52}$$

These fields are tangent to the net and moreover, Eq. (17) takes especially simple form

$$\vec{W}_{,u} = p\vec{Z}, \qquad \vec{Z}_{,v} = q\vec{W}. \tag{53}$$

These are equations of linear problem of bi-component KP hierarchy what justifies our interest.

As in continuous case we introduce *the discrete canonical tangent fields* $\vec{W}$ and $\vec{Z}$ of the discrete asymptotic net

$$\vec{R}_{(12)} - \vec{R}_{(2)} = \alpha \vec{W}, \qquad \vec{R}_{(12)} - \vec{R}_{(1)} = \beta \vec{Z}, \tag{54}$$

where functions α and β are defined by

$$\beta_{(2)} = \frac{B_{(1)}}{H}\beta, \qquad \alpha_{(1)} = \frac{A_{(2)}}{H}\alpha. \tag{55}$$

Now Eq. (20) in terms of fields $\vec{W}$ and $\vec{Z}$ are

$$\Delta_{(1)}\vec{W} = \mathcal{P}\vec{Z}, \qquad \Delta_{(2)}\vec{Z} = \mathcal{Q}\vec{W}, \tag{56}$$

where

$$\mathcal{P} = \frac{P_{(2)}B_{(1)}}{A_{(2)}}\frac{\beta}{\alpha}, \qquad \mathcal{Q} = \frac{Q_{(1)}A_{(2)}}{B_{(1)}}\frac{\alpha}{\beta}. \tag{57}$$

9. Jonas formulas

In 1920, Jonas [27] published a supplement of the theory of transformations of asymptotic nets. It allowed him to find a few system of differential equations which posses the Darboux–Bäcklund transformation and superposition principle for solutions [27–29]. Here, we recall the Jonas formulas and we discretize them in the next section.

Let an asymptotic net is given by Lelieuvre representation $\hat{N}$ (23) and (24). We have the natural basis $\{\hat{N}; \hat{N}_{,u}; \hat{N}_{,v}\}$ at each point of the net. Let us split the Moutard transform $\hat{N}'$ of $\hat{N}$ into components with respect to the basis

$$\Theta\hat{N}' = x^1\hat{N}_{,u} + x^2\hat{N}_{,v} + x^3\hat{N}. \tag{58}$$

On inserting (58) into the Moutard transformation (28) due to fact that $\{\hat{N}; \hat{N}_{,u}; \hat{N}_{,v}\}$ is a basis the coefficients x^i have to satisfy the following system:

$$\frac{\partial}{\partial u}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\vartheta_{,u} & 0 & -1 \\ p & 0 & 0 \\ -(p_{,v} + p\vartheta_{,v}) & -(pq + \vartheta_{,uv}) & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -\Theta \\ 0 \\ \Theta_{,u} \end{bmatrix},$$

$$\frac{\partial}{\partial v}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & q & 0 \\ 0 & -\vartheta_{,v} & -1 \\ -(pq + \vartheta_{,uv}) & -(q_{,u} + q\vartheta_{,u}) & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \Theta \\ -\Theta_{,v} \end{bmatrix}. \tag{59}$$

The above system is compatible *iff* functions p, q and ϑ satisfy the differential constraint (18) and $\Theta_{,uv} = (pq + \vartheta_{,uv})\Theta$ (compare (30)). The covector $\hat{N}'$ satisfies Eq. (25), but with new functions p', q' and ϑ' which are related to the old ones via

$$p' = -p + x_2\frac{s}{\Lambda}, \qquad q' = -q + x_1\frac{t}{\Lambda}, \qquad \mathrm{e}^{\vartheta'} = \mathrm{const.}\,\frac{\mathrm{e}^{\vartheta}\Lambda}{\Theta^2}, \tag{60}$$

where

$$s=\Theta_{,uu}-a\Theta_{,u}+p\Theta_{,v}-(pb+p_{,v})\Theta,$$
$$t=\Theta_{,vv}+q\Theta_{,u}-b\Theta_{,v}-(qa+q_{,u})\Theta,$$
$$\Lambda=x_1\Theta_{,u}+x_2\Theta_{,v}+x_3\Theta. \tag{61}$$

Direct calculations show that

$$\Lambda_{,u}=x^1 s,\qquad \Lambda_{,v}=x^2 t, \tag{62}$$

$$t_{,u}=qs,\qquad s_{,v}=pt. \tag{63}$$

As it was shown by Jonas [27] to construct a second focal surface of W-congruence from the first one it is sufficient to find a solution of

$$x_{2,u}=px_1,\qquad x_{1,v}=qx_2 \tag{64}$$

these are the two of Eqs. (59). Let us draw your attention that in fact we have to find an additional solution of the equations of tangent canonical fields.

10. Discrete Jonas formulas

We extend the consideration of the previous section on the discrete level. Let $\hat{N}'$ be a transform of $\hat{N}$ under the discrete Moutard transformation. On decomposing $\hat{N}'$ with respect to the basis $\{\hat{N};\hat{N}_{(1)};\hat{N}_{(2)}\}$

$$\Theta\hat{N}'=x^1\hat{N}_{(1)}+x^2\hat{N}_{(2)}+x^3\hat{N}, \tag{65}$$

we obtain after substitution into discrete Moutard transformation that coefficient $\{x_i\}$ satisfies the following system (analogue of (59)):

$$x^1_{(1)}=-\frac{1}{A}\left(\frac{1}{F}x^2+x^3+\Theta_{(1)}\right),\qquad x^3_{(1)}+(\gamma+1+A+P)x^1_{(1)}=x^1-x^2-\Theta,$$
$$x^2_{(1)}-Px^1_{(1)}=\frac{1}{F}x^2, \tag{66}$$

$$x^2_{(2)}=-\frac{1}{B}\left(\frac{1}{F}x^1+x^3-\Theta_{(2)}\right),\qquad x^3_{(2)}+(\delta+1+B+Q)x^1_{(2)}=x^2-x^1+\Theta,$$
$$x^1_{(2)}-Qx^2_{(2)}=\frac{1}{F}x^1. \tag{67}$$

One can show that covector $\hat{N}'$ satisfies the primed analogue of (46) and primed functions are related to non-primed ones via

$$P'=\left(-P+\frac{S}{L}\frac{x^2}{F}\right)\frac{\Theta_{(12)}}{\Theta_{(11)}},\qquad Q'=\left(-Q+\frac{T}{L}\frac{x^1}{F}\right)\frac{\Theta_{(12)}}{\Theta_{(22)}},$$
$$A'=A\left(1+\frac{S}{L}x^1_{(1)}\right)\frac{\Theta}{\Theta_{(11)}},\qquad B'=B\left(1+\frac{T}{L}x^2_{(2)}\right)\frac{\Theta}{\Theta_{(22)}}, \tag{68}$$

where

$$S=\Theta_{(11)}-(\gamma+1+A+P)\Theta_{(1)}+A\Theta+P\Theta_{(12)},$$
$$T=\Theta_{(22)}-(\delta+1+B+Q)\Theta_{(2)}+B\Theta+Q\Theta_{(12)},$$
$$L=x^1\Theta_{(1)}+x^2\Theta_{(2)}+x^3\Theta. \tag{69}$$

Direct calculations shows that (analogue of (62))

$$\Delta_{(1)}L=x^1_{(1)}S, \qquad \Delta_{(2)}L=x^2_{(2)}T, \tag{70}$$

$$T_{(1)}=F_{(2)}T+Q_{(1)}F_{(1)}S, \qquad S_{(2)}=F_{(1)}S+P_{(2)}F_{(2)}T. \tag{71}$$

On defining α^{II} and β^{II} via

$$\beta^{\mathrm{II}}_{(1)}=F_{(2)}\beta^{\mathrm{II}}, \qquad \alpha^{\mathrm{II}}_{(2)}=F_{(1)}\alpha^{\mathrm{II}}, \tag{72}$$

and introducing

$$\mathbb{S}:=\frac{S}{\alpha^{\mathrm{II}}}, \qquad \mathbb{T}:=\frac{T}{\beta^{\mathrm{II}}}, \tag{73}$$

Eq. (71) take the form (analogue of (63))

$$\Delta_{(1)}\mathbb{T}=\frac{Q_{(1)}F_{(1)}}{F_{(2)}}\frac{\alpha^{\mathrm{II}}}{\beta^{\mathrm{II}}}\mathbb{S}, \qquad \Delta_{(2)}\mathbb{S}=\frac{P_{(2)}F_{(2)}}{F_{(1)}}\frac{\beta^{\mathrm{II}}}{\alpha^{\mathrm{II}}}\mathbb{T}. \tag{74}$$

Let us rewrite the third equation of (66) and (67) in the form

$$x^1_{(12)}=\frac{1}{HF_{(1)}}x^1_{(1)}+\frac{Q_{(1)}}{HF_{(2)}}x^2_{(2)}, \qquad x^2_{(12)}=\frac{1}{HF_{(2)}}x^2_{(2)}+\frac{P_{(2)}}{HF_{(1)}}x^1_{(1)}. \tag{75}$$

On introducing

$$w:=\frac{x^1_{(1)}}{\alpha^{\mathrm{I}}}, \qquad z:=\frac{x^2_{(2)}}{\beta^{\mathrm{I}}}, \tag{76}$$

where functions α^{I} and β^{I} are defined by

$$\beta^{\mathrm{I}}_{(1)}=\frac{1}{HF_{(2)}}\beta^{\mathrm{I}}, \qquad \alpha^{\mathrm{I}}_{(2)}=\frac{1}{HF_{(1)}}\alpha^{\mathrm{I}}, \tag{77}$$

we can write Eq. (75) in the form

$$\Delta_{(1)}z=\frac{P_{(2)}F_{(2)}}{F_{(1)}}\frac{\alpha^{\mathrm{I}}}{\beta^{\mathrm{I}}}w, \qquad \Delta_{(2)}w=\frac{Q_{(1)}F_{(1)}}{F_{(2)}}\frac{\beta^{\mathrm{I}}}{\alpha^{\mathrm{I}}}z. \tag{78}$$

It is easy to see that owing to fact that β^{I} and α^{I} are given up to multiplication by function of single variable, i.e. $\alpha^{\mathrm{I}}\to\alpha^{\mathrm{I}}g^{\mathrm{I}}(m_2)$ and $\beta^{\mathrm{I}}\to\beta^{\mathrm{I}}f^{\mathrm{I}}(m_1)$ (see (77)), Eq. (78) can be put in the form

$$\Delta_{(1)}z=\mathcal{P}w, \qquad \Delta_{(2)}w=\mathcal{Q}z. \tag{79}$$

Indeed, it is enough to observe that

$$\diamond\left(\frac{\mathcal{P}}{(P_{(2)}F_{(2)}/F_{(1)})(\alpha^{\mathrm{I}}/\beta^{\mathrm{I}})}\right)=1\Rightarrow\left(\frac{\mathcal{P}}{(P_{(2)}F_{(2)}/F_{(1)})(\alpha^{\mathrm{I}}/\beta^{\mathrm{I}})}\right)=U^{\mathrm{I}}(m_1)V^{\mathrm{I}}(m_2),$$

$$\diamond\left(\frac{\mathcal{Q}}{(Q_{(1)}F_{(1)}/F_{(2)})(\beta^{\mathrm{I}}/\alpha^{\mathrm{I}})}\right)=1\Rightarrow\left(\frac{\mathcal{Q}}{(Q_{(1)}F_{(1)}/F_{(2)})(\beta^{\mathrm{I}}/\alpha^{\mathrm{I}})}\right)=U^{\mathrm{II}}(m_1)V^{\mathrm{II}}(m_2),$$

$$\mathcal{P}\mathcal{Q}=P_{(2)}Q_{(1)}=\frac{P_{(2)}F_{(2)}}{F_{(1)}}\frac{Q_{(1)}F_{(1)}}{F_{(2)}}\Rightarrow\frac{\mathcal{P}\mathcal{Q}}{P_{(2)}Q_{(1)}}=\frac{U^{\mathrm{I}}(m_1)V^{\mathrm{I}}(m_2)}{U^{\mathrm{II}}(m_1)V^{\mathrm{II}}(m_2)}=1, \qquad (80)$$

hence on setting $g^{\mathrm{I}}(m_2)=1/V^{\mathrm{I}}(m_2)$ and $f^{\mathrm{I}}(m_1)=U^{\mathrm{I}}(m_1)$, we obtain (79). Therefore functions w and z and components of discrete canonical fields satisfy the same equation (compare Eqs. (56) and (79)) just as in the continuous case function x_1 and x_2 and components of canonical fields satisfy the same equation (compare Eqs. (53) and (64)).

11. The discrete Fubini–Ragazzi system

We apply results from the previous section to obtain a discrete version of the Fubini–Ragazzi system (1). Namely, one can interpret the equations of the tangent canonical fields (53) of an asymptotic net and the discrete tangent canonical fields (53) of a discrete asymptotic net as equations of tangent vectors of two-dimensional conjugate net and quadrilateral lattice, respectively (see, e.g. [17,18]). Therefore one can apply theory of reductions of developed for conjugate nets [49,50] and quadrilateral lattices [10,13,17,32]. In the recent paper [17] Doliwa and Santini have introduced the notion of symmetric reductions of discrete multidimensional quadrilateral lattices.

Since symmetric reduction (see [17]) imposed on the tangent canonical fields (53) of an asymptotic net, i.e.

$$\left(\log\frac{p}{q}\right)_{,uv}=0$$

together with compatibility conditions of the asymptotic net (18) gives Fubini–Ragazzi system (1) we conclude that symmetric reduction (see Proposition 4.8 of paper [17]) imposed on the discrete tangent canonical fields (56) of a discrete asymptotic net

$$\diamond\frac{\mathcal{P}}{\mathcal{Q}}=\frac{H_{(2)}}{H_{(1)}}$$

together with Eq. (22) gives the discrete version of the Fubini–Ragazzi system (6). The Darboux–Bäcklund transformation of the discrete Fubini–Ragazzi system was established by us so far. Since we are preparing the extended exposition of the Fubini–Ragazzi system and its discrete version we are hanging up the discussion at this moment.

As we have suggested the theory of reductions of the lattice Darboux system [8], which was applied primarily for quadrilateral lattices [10,13,17,32], one can extend for theory of discrete asymptotic nets. In particular, others classical integrable systems [24,27,28] can be discretized in this way.

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